Birzeit University
Mathematics Department
Math 1321

First Exam — Second Semester 2016/2017

Student Name (**IN ARABIC**):................................Number:................................

Name of discussion teacher: ....Section:................................

This Exam consists of 7 pages. — Exam Time 90 Minutes

Question 1 (60%). Circle the most correct answer

1. $\lim\limits_{n\to\infty} \frac{4+(-1)^n}{n} =$
   (a) 5
   (b) 0
   (c) 3
   (d) DNE

2. The sequence $\{a_n\}$, where $a_n = \frac{2n+1}{6n+1}$ is
   (a) nondecreasing and not bounded.
   (b) nonincreasing and bounded.
   (c) nondecreasing and bounded.
   (d) nonincreasing and not bounded.

3. The sequence $\left\{(-1)^n\left(1-\frac{6}{n}\right)\right\}$
   (a) converges to 1
   (b) converges to -1
   (c) converges to 0
   (d) diverges

4. $\sum\limits_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$
   (a) converges
   (b) diverges

5. If $a_n > 0$ and $b_n > 0$ for all $n > N$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then
   (a) The series $\sum a_n$ and $\sum b_n$ both converge or both diverge.
   (b) If $\sum b_n$ diverges, then $\sum a_n$ diverges.
   (c) If $\sum b_n$ converges, then $\sum a_n$ converges.
   (d) If $\sum a_n$ converges, then $\sum b_n$ converges.

6. The sum of the series $\sum_{n=0}^{\infty}(\frac{1}{4^n} - \frac{(-1)^n}{4^{n+1}})$ is

(a) $\frac{32}{15}$
(b) $\frac{2}{15}$
(c) $\frac{17}{15}$
(d) $\frac{8}{15}$

7. The series $\sum_{n=1}^{\infty} \frac{n}{(\ln n+10)^n}$

(a) converges by nth term test
(b) diverges by nth term test
(c) diverges by nth root test
(d) converges by nth root test

8. For the series $\sum_{n=1}^{\infty} \frac{5}{n(n+1)}$

(a) $s_n = \frac{5n}{n+1}$, sum $= 5$
(b) $s_n = \frac{5(n+1)}{n}$, sum $= 5$
(c) $s_n = \frac{5n-5}{n}$, sum $= 5$
(d) $s_n = \frac{5n+10}{n+1}$, sum $= 5$

9. For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x+6)^n}{\sqrt{n}}$ converges conditionally

(a) $x = -5, x = -7$
(b) $x = -5$
(c) $x = -7$
(d) none

10. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{5}{4}}+3}$

(a) converges conditionally
(b) converges absolutely
(c) diverges conditionally
(d) diverges

11. The series $\sum_{n=1}^{\infty} a_n$, where $a_1 = 4$, $a_{n+1} = \sqrt[n]{n}\, a_n$

(a) converges
(b) diverges

12. The series $\sum_{n=1}^{\infty} \frac{5n}{n^2+1}$

(a) converges by direct comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(b) converges by limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(c) diverges by nth term test

(d) diverges by limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$

13. If $0 \leq a_n \leq b_n$, for all $n \geq 1$. Then one of the following statements is true

(a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

(b) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ converges.

(c) If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

(d) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

14. The series $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{1.3.5 \cdots (2n-1)}{(2n-1)!}$

(a) converges conditionally

(b) converges absolutely

(c) diverges

(d) diverges absolutely

15. The sequence $a_1 = 3$, $a_{n+1} = \frac{n}{n+1} a_n$

(a) converges to 3

(b) converges to 1

(c) converges to 0

(d) diverges

16. The series $\sum_{n=1}^{\infty} e^{-n}$

(a) diverges

(b) converges to $\frac{1}{e-1}$

(c) converges to $\frac{e}{e-1}$

(d) converges to $\frac{1}{e}$

17. The interval of convergence I of the series $\sum_{n=1}^{\infty} \frac{(x+4)^n}{n3^n}$ is

(a) $I = (-7, -1]$
(b) $I = [-7, -1)$
(c) $I = [-7, -1]$
(d) $I = (-7, -1)$

18. Consider the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$, $u_n > 0$. if $\lim_{n\to\infty} u_n \neq 0$, then

(a) the series converges
(b) the series diverges
(c) the series may or may not converge
(d) None

19. $1.\overline{7} =$

(a) $\frac{70}{9}$
(b) $\frac{7}{9}$
(c) $\frac{16}{9}$
(d) this number is not a rational number

20. If the first three terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}}$ are used to estimate its sum, then the error E satisfies

(a) $\frac{-1}{32} \leq E \leq 0$
(b) $0 \leq E \leq \frac{1}{32}$
(c) $\frac{-1}{16} \leq E \leq 0$
(d) $0 \leq E \leq \frac{1}{16}$

Bonus $(2^{2^n})^2 =$

(a) 2^{4^n}
(b) $2^{2^{n+1}}$
(c) $2^{2^{2n}}$
(d) $2^{2^{n^2}}$

Question 2 ((7+10)%). (a) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n 3n^2}{n^3+1}$ converge absolutely, converge conditionally, or diverge?

S.O.A.V is $\sum_{n=1}^{\infty} \frac{3n^2}{n^3+1}$ diverges by L.C.T. with $\sum_{n=1}^{\infty} \frac{1}{n}$ which div.

$\lim_{n\to\infty} \frac{\frac{3n^2}{n^3+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n^3}{n^3+1} = 3.$

so $\sum_{n=1}^{\infty} \frac{3n^2}{n^3+1}$ div.

Now by ~~Alsen~~ Alternating series test on $\sum \frac{(-1)^n 3n^2}{n^3+1}$

① $u_n = \frac{3n^2}{n^3+1} > 0, \forall n > 1$

② $u_{n+1} \le u_n, \forall n \geq 2$ since $f(x) = \frac{3x^2}{x^3+1}$ is decreasing

$f'(x) = \frac{3x(2-x^3)}{(x^3+1)^2} < 0$ for all $x \geq 2$

③ $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{3n^2}{n^3+1} = 0$

(b) For what values of x does the series $\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$ converges absolutely.

Ratio test on $\sum_{n=2}^{\infty} \left|\frac{x^n}{n(\ln n)^2}\right| \Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)(\ln(n+1))^2} \cdot \frac{n(\ln n)^2}{x^n} \right|$

$= \lim_{n\to\infty} |x| \left(\frac{n}{n+1} \cdot \frac{(\ln n)^2}{(\ln(n+1))^2} \right)$

$= |x| \cdot 1 \cdot 1 = |x|.$

~~so~~ conv. abs. if $|x| < 1$.

End-points: ① $x = 1$ the series is $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

which converges absolutely by integral test

$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \frac{1}{\ln x}\Big|_2^b = \lim_{b\to\infty} \left(\frac{1}{\ln b} - \frac{1}{\ln 2}\right) = -\ln 2$

and $f(x) = \frac{1}{x(\ln x)^2}$ is decreasing and positive.

② At $x = -1$, the series is $\sum \frac{(-1)^n}{n(\ln n)^2}$ which converges absolutely as in ①

so $\sum \frac{x^n}{n(\ln n)^2}$ converges absolutely for $-1 \le x \le 1$.

Question 3 (13%). consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^{n+1}}{n4^n}$

(a) Find the interval and radius of convergence

By Radion test, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+2}(x-2)^{n+2}}{(n+1)4^{n+1}} \cdot \frac{n\,4^n}{(-1)^{n+1}(x-2)^{n+1}}\right|$

$= \lim_{n\to\infty} \left|\frac{(x-2)}{4} \cdot \frac{n}{n+1}\right| = \frac{|x-2|}{4}$

converges abs. for $\left|\frac{x-2}{4}\right| < 1 \iff -2 < x < 6$

Endpoints: ① $x=6 \to$ The series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4^{n+1}}{n 4^n} = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n}$

which converges conditionally (Alternating harmonic series).

② $x=-2 \Rightarrow$ The series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-4)^{n+1}}{n4^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+2} 4^{n+1}}{n \cdot 4^n}$

$= \sum_{n=1}^{\infty} \frac{4}{n}$ which diverges.

(b) For what values of x does the series converge absolutely

so interval of convergence is $-2 < x \leq 6$

radius of convergence $R = 4$.

The series converges absolutely on $-2 < x < 6$

(c) For what values does it converge conditionally

The series conv. cond. at $x = 6$.

Question 4 (10%). Find the radius and interval of convergence of the series $\sum_{n=0}^{\infty} (\log_2 x)^n$.

This is a geometric series, $r = \log_2 x$

converges abs. for $|r| < 1 \iff |\log_2 x| < 1$

$$-1 < \log_2 x < 1$$

$$\iff \quad \frac{-1}{2} < x < 2 \,, \quad \text{diverges if } x \geq 2 \text{ or } x \leq 0.5$$

interval of convergence $0.5 < x < 2$

radius of conv. $R = \frac{3}{4}$.

Bonus: Find the sum of the series $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$ (Hint: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for $-1 < x < 1$)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \,, \quad -1 < x < 1$$

Differentiate both sides to get

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n\, x^{n-1}$$

At $x = \frac{1}{2} \Rightarrow \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n-1} = \sum n \frac{1}{2^{n-1}} = \frac{1}{(1-\frac{1}{2})^2} = \frac{1}{1/4}$

$= 4.$

So $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = 4.$